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ON THE CLASSIFICATION OF $(n-k+1)$-CONNECTED EMBEDDINGS OF n-MANIFOLDS INTO $(n+k)$-MANIFOLDS IN THE METASTABLE RANGE

LIU RONG

ABSTRACT. For an $(n-k+1)$-connected map f from a connected smooth n-manifold M to a connected smooth $(n+k)$-manifold V, where M is closed, we work out the isotopy group $[M \subset V]_f$ in the metastable range $n \leq 2k-4$. To prove our results, we develop the Hurewicz-type theorems which provide us with the efficient methods of computing the homology groups with local coefficients from the homotopy groups.

0. INTRODUCTION

Let M^n and V^{n+k} be connected smooth manifolds of dimensions n and $n+k$ and $f: M \to V$ a smooth map. Assume that M is closed. Designate $[M \subset V] = \pi_1(V^M, \operatorname{Emb}(M, V), f)$ the set of isotopy classes of embeddings with a specific homotopy to f where V^M means the space of smooth maps from M to V and $\operatorname{Emb}(M, V)$ is the subspace of smooth embeddings of M in V. It is well known that $[M \subset V]_f$ is an abelian affine group in the metastable range $n \leq 2k-4$ and is called the isotopy group (cf. [11]).

Suppose that $f: M \to V$ is $(n-k+1)$-connected. A theorem due to Haefliger [6] asserts that f is homotopic to an embedding for $n \leq 2k-3$. In this case the set $[M \subset V]_f$ is nonempty and it is meaningful to enumerate it. Without loss of generality, we assume that $f: M^n \to V^{n+k}$ is an $(n-k+1)$-connected embedding and we identify M with $f(M) \subset V$. Then our results could be stated as follows.

0.1. **Theorem.** *Let $f: M^n \to V^{n+k}$ be an $(n-k+1)$-connected embedding. If $n \leq 2k-4$, $k < n$, then*

$$[M^n \subset V^{n+k}]_f = \begin{cases} H_{n-k+2}(V, M; Z_2) & \text{if } k \text{ is even}, \\ H_{n-k+2}(V, M; Z_V) & \text{if } k \text{ is odd}, \end{cases}$$

where Z_V is the orientation local system of manifold V.

This theorem generalizes the main result of A. Haefliger and M. Hirsch [8]; in the case that $V^{n+k} = R^{n+k}$, it can be deduced from the result of N. Habegger [4] as well.

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In the following theorems we denote by $\pi_1^+(M)$ the subgroup of $\pi_1(M)$ whose elements are represented by the orientation-preserving loops of M. Let f_π be the homomorphism $f_*\colon \pi_1(M) \to \pi_1(V)$ induced by f.

0.2. **Theorem.** *Let $f\colon M^n \to V^{2n}$ be a 1-connected embedding. Suppose that $\ker f_\pi \subseteq \pi_1^+(M)$ and $n \geq 4$. Then*

$$[M^n \subset V^{2n}]_f = \begin{cases} H_2(V, M; Z_2) & \text{if } n \text{ is even}, \\ H_2(V, M; Z_V) & \text{if } n \text{ is odd}, \end{cases}$$

where Z_V is the orientation sheaf of manifold V.

0.3. **Theorem.** *Let $f\colon M^n \to V^{2n}$ be a 1-connected embedding, where $n \geq 4$. Suppose that $\ker f_\pi \not\subseteq \pi_1^+(M)$ and $w_1(V) = 0$. Then*

$$[M^n \subset V^{2n}]_f = \begin{cases} Z \oplus H_2^{(+)}(V, M; Z_2) & \text{if } n \text{ is even}, \\ H_2(V, M; Z_2) & \text{if } n \text{ is odd}, \end{cases}$$

where $H_2^{(+)}(V, M; Z_2)$ is the kernel of composition

$$H_2(V, M; Z_2) \xrightarrow{\partial} H_1(M; Z_2) \xrightarrow{w_1(M)} Z_2.$$

The above theorems generalize the results of A. Haefliger [7] in the case that $V^{2n} = R^{2n}$.

Now assume V is a nonorientable manifold and denote by $p\colon \overline{V} \to V$ its orientation double covering. Let $T\colon \overline{V} \to \overline{V}$ be the nontrivial covering transformation of p. Set $\overline{M} = p^{-1}(M)$. (Notice that in general $\overline{M}$ is not the orientation covering of M.) We have

0.4. **Theorem.** *Let $f\colon M^n \to V^{2n}$ be a 1-connected embedding, where $n \geq 4$. Suppose that $\ker f_\pi \not\subseteq \pi_1^+(M)$ and $w_1(V) \neq 0$. Then there is an exact sequence*

$$0 \to \frac{H_2^{(+)}(\overline{V}, \overline{M}; Z_2)}{\langle x + T_*(x)\colon x \in H_2^{(+)}(\overline{V}, \overline{M}; Z_2)\rangle} \to [M^n \subset V^{2n}]_f \to Z_2 \to 0 \quad \text{if } n \text{ is even},$$

and an isomorphism

$$[M^n \subset V^{2n}]_f \approx H_2(V, M; Z_2) \quad \text{if } n \text{ is odd}.$$

In this paper we refer to the singularity approaches [2, 9, 12, 13] which convert the enumeration of $[M^n \subset V^{n+k}]_f$ into the calculation of

$$H_{n-k+1}(\Lambda_f \times_2 S^\infty, M \times P^\infty; Z_{\Psi(f)})$$

for an $(n-k+1)$-connected map f. In §1, we recall the definition of the topological space $\Lambda_f \times_2 S^\infty$ and discuss its homotopy properties. Our Theorem 0.1 is proved in §2 by using a relative twisted Hurewicz theorem—Proposition 2.1. Sections 3–5 are engaged in the proof of our Theorems 0.2–0.4. The following twisted Hurewicz theorem is established at the beginning of §3 and afterwards showed very powerful in our computations of the first homology groups with local coefficients.

3.1. **Theorem.** *Let X be a path-connected topological space. Suppose that Z_ϕ is a local system of integers on X characterized by a homomorphism $\phi\colon \pi_1(X) \to$ Aut Z. Then there is an isomorphism*

$$H_1(X; Z_\phi) \cong \pi_1^+/([\pi_1^+, \pi_1^+]\cdot[\pi_1^-]^2),$$

where $\pi_1^+ = \ker\phi$, $\pi_1^- = \pi_1(X)\backslash\pi_1^+$, $[\pi_1^+, \pi_1^+]$ is the commutative group of π_1^+, $[\pi_1^-]^2$ is the normal subgroup of π_1^+ generated by the elements x^2 for $x \in \pi_1^-$.

1. Preliminaries

Let $f\colon M^n \to V^{n+k}$ be an embedding. Denote by $\Lambda_f = P(V; M, M)$ the space of paths in V from M to M. Naturally, there is an inclusion $M \subset P(V; M, M)$ induced by the constant paths of $M \subset V$. Let S^∞ be the unit sphere in an infinite-dimensional Hilbert space. Designate $\Lambda_f \times_2 S^\infty$ as the quotient of the product $\Lambda_f \times S^\infty$ by the involution $(\sigma, \alpha) \to (\sigma^{-1}, -\alpha)$. Certainly, $M \times P^\infty$ (the quotient of $M \times S^\infty$) is a subspace of $\Lambda_f \times_2 S^\infty$.

Let τM and τV be vector bundles over $\mathfrak{S}_2\tau M = (\tau M \times \tau M) \times_2 S^\infty$ and $\mathfrak{S}_2 M = (M \times M) \times_2 S^\infty$, respectively. Hence there is a virtual bundle $\Psi(f) = p_1^*(\mathfrak{S}_2\tau M) - p_2^*(\tau V \tilde{\otimes} \lambda) \oplus \varepsilon^{n-k}$ over $\Lambda_f \times_2 S^\infty$ where $p_1\colon \Lambda_f \times_2 S^\infty \to \mathfrak{S}_2 M$ and $p_2\colon \Lambda_f \times_2 S^\infty \to V \times P^\infty$ are given by $p_1([\sigma, \alpha]) = [\sigma(-1), \sigma(1), \alpha]$ and $p_2([\sigma, \alpha]) = (\sigma(0), [\alpha])$. Summing up the results of [2, 9, 12, 13], we have

1.1. **Proposition.** *Let $f\colon M^n \to V^{n+k}$ be an embedding, $n \le 2k-4$. Then there is a bijection $\alpha\colon [M \subset V]_f \to \Omega_{n-k+1}(\Lambda_f \times_2 S^\infty, M \times P^\infty; \Psi(f))$.*

1.2. **Proposition.** *There is a natural isomorphism*

$$A\colon \pi_p(P(V; M, M), M, *) \to \pi_{p+1}(V, M, *)$$

compatible with the actions of $\pi_1(M, *)$ on them for $p \ge 1$.

Proof. Set $\pi_p(P(V; M, M), M, *) = [D^p, S^{p-1}, s; P(V, M, M), M, *]$. The exponential law asserts that $\pi_p(P(V; M, M), M, *)$ is in one-one correspondence with the homotopy classes of maps $\phi\colon I \times D^p, \partial(I \times D^p) \to V, M$ such that $\phi(t, s) = *$ and $\phi(t, x) = \phi(0, x) = \phi(1, x)$ for $t \in I$, $x \in S^{p-1}$. It is clear that D^{p+1} is homeomorphic to the quotient space of $I \times D^p$ in which (t, x) is identified with $(0, x)$ for each $(t, x) \in I \times S^{p-1}$. ϕ can be factored by $\overline{\phi}\colon D^{p+1}, S^p, s \to V, M, *$. It follows that there is a 1-1 correspondence A between $\pi_p(P(V; M, M), M, *)$ and $\pi_{p+1}(V, M, *)$. Because the above operation is compatible with the Co-H structures of D^p and D^{p+1} and their S^1-coproducts (cf. [14, pp. 45–51]), A is an isomorphism commuting with the actions of $\pi_1(M, *)$. Q.E.D.

1.3. **Proposition.** *If $f\colon M^n \to V^{n+k}$ is $(n-k+1)$-connected for $n \le 2k-4$, then there is a bijection*

$$\alpha'\colon [M \subset V]_f \to H_{n-k+1}(\Lambda_f \times_2 S^\infty; M \times P^\infty; Z_{\Psi(f)}),$$

where $Z_{\Psi(f)}$ is the local system on $\Lambda_f \times_2 S^\infty$ associated with $w_1(\Psi(f))$.

Proof. Observe that $(P(V; M, M) \times S^\infty, M \times S^\infty)$ is a double covering of $(\Lambda_f \times_2 S^\infty, M \times P^\infty)$. From Proposition 1.2, that $(\Lambda_f \times_2 S^\infty, M \times P^\infty)$ is $(n-k)$-connected. By Proposition 5.1 of [2], there is an isomorphism

$\mu\colon \Omega_{n-k+1}(\Lambda_f \times_2 S^\infty, M \times P^\infty; \Psi(f)) \to H_{n-k+1}(\Lambda_f \times_2 S^\infty, M \times P^\infty; Z_{\Psi(f)})$. Hence α' is obtained as $\mu \circ \alpha$, where α is given in Proposition 1.1. Q.E.D.

To calculate the homology group in the above proposition, we should first determine the homomorphism $w_1(\Psi(f))\colon \pi_1(\Lambda_f \times_2 S^\infty) \to Z_2$.

1.4. **Lemma.** *$\pi_1(P(V; M, M, *)$ is isomorphic to the semi product $\pi_2(V, M, *) \times_h \pi_1(M, *)$ where h is the action of $\pi_1(M, *)$ on $\pi_2(V, M, *)$.*

Proof. Set $\pi_1(P(V; M, M), *) = [I, \dot{I}; P(V; M, M), *]$. The exponential law gives a one-one correspondence

$$\pi_1(P(V; M, M), *) \leftrightarrow [J \times I, \dot{J} \times I, J \times \dot{I}; V, M, M, *].$$

Since $J \times I$ can be identified with the oriented $I \times I$, every

$$[g] \in [J \times I, \dot{J} \times I, J \times \dot{I}; V, M, M, *]$$

naturally determines an element of

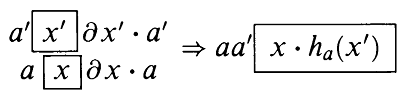
$$[g]' \in [J \times I, \partial(J \times I), (-1, 0); V, M, *] = \pi_2(V, M)$$

and an element $[g|_{-1\times I}] \in \pi_1(M)$. Conversely, for every

$$[g] \in [J \times I, \partial(J \times I), (-1, 0); V, M, *] \quad \text{and} \quad [\alpha] \in \pi_1(M)$$

we can first rearrange $\partial g = g|_{\partial(J\times I)}$ by a homotopy to a map $\dot{g}'\colon \partial(J \times I) \to M$ such that $\dot{g}'|_{-1\times I} = \alpha$, $\dot{g}'|_{1\times I} = \partial g * \alpha$, and $\dot{g}'(J \times \dot{I}) = *$. The homotopy extension property shows that g is homotopic to g' such that $\partial g' = \dot{g}'$. The homotopy $[g'] \in [J \times I, \dot{J} \times I, J \times \dot{I}; V, M, M, *]$ is uniquely determined by $[g] \in \pi_2(V, M)$ and $[\alpha] \in \pi_1(M)$.

It follows that there is a one-one correspondence between $\pi_1(P(V; M, M))$ and $\pi_2(V, M) \times \pi_1(M)$. Observing the following figure,



we know that $\pi_1(P(V; M, M))$ induces a product on $\pi_2(V, M) \times \pi_1(M)$ as $(x, a) \cdot (x', a') = (x \cdot h_a(x'), aa')$ and the lemma is valid. Q.E.D.

1.5. **Proposition.** *There is an isomorphism*

$$\pi_1(\Lambda_f \times_2 S^\infty) \cong (\pi_2(V, M) \times_h \pi_1(M)) \times_\phi T_2,$$

where T_2 is the multiplicative group of two elements 1 and m, ϕ is the action of T_2 on $\pi_2(V, M) \times_h \pi_1(M)$ given by $\phi(m)(x, a) = (x^{-1}, \partial x \cdot a)$.

Proof. Since the double covering $P(V; M, M) \times S^\infty \to \Lambda_f \times_2 S^\infty$ induces a partially split exact sequence $1 \to \pi_1(P(V; M, M)) \to \pi_1(\Delta_f \times_2 S^\infty) \to T_2 \to 1$ and the involution $T\colon P(V; M, M) \to P(V; M, M)$ defined by $T(\sigma) = \sigma^{-1}$ gives the semiproduct $\pi_1(\Delta_f \times_2 S^\infty) \approx \pi_1(P(V; M, M) \times_\phi T_2$, this proposition follows from Lemma 1.4. Q.E.D.

Recall from [11] that $\pi_1(\mathfrak{S}_2 M) = (\pi_1(M) \times \pi_1(M)) \times_\phi T_2$ where ϕ is the action of T_2 on $\pi_1(M) \times \pi_1(M)$ given by $\phi(m)(a, b) = (b, a)$. Consider the figure in the proof of Lemma 1.1. The fibration $p_1\colon \Lambda_f \times_2 S^\infty \to \mathfrak{S}_2 M$ induces $p_{1\pi}\colon \pi_1(\Lambda_f \times_2 S^\infty) \to \pi_1(\mathfrak{S}_2 M)$ given by $p_{1\pi}((x, a, 1)) = (a, \partial x \cdot a; 1)$ and $p_{1\pi}((x, a, m)) = (a, \partial x \cdot a; m)$. From Proposition 2.3 in [12], it follows that

1.6. **Proposition.** *The local system* $Z_{\Psi(f)}$ *is determined by a homomorphism*

$$\Psi: \pi_1(\Lambda_f \times_2 S^\infty) \to \operatorname{Aut} Z$$

such that

$$\Psi((x, a, 1)) = (-1)^{\partial x}(-1)^{f_\pi(a)},$$
$$\Psi((x, a, m)) = (-1)^k(-1)^{\partial x}(-1)^{f_\pi(a)} \quad \textit{for } x \in \pi_2(V, M), \; a \in \pi_1(M)$$

where $(-1)^{\partial x} = (-1)^{w_1(M)[\partial x]}$, $(-1)^{f_\pi(a)} = (-1)^{w_1(V)[f_\pi(a)]}$.

2. Proof of Theorem 0.1

First of all, we discuss a generalization of relative Hurewicz theorem.

Let (X, Y) be a pair of topological spaces. For convenience, we assume that X and Y are path connected and locally path connected and that $\pi_1(X, *) = \pi_1(Y, *)$. Then we have

2.1. **Proposition.** *Let* A_ϕ *be a local system on* X *characterized by a right action* $\phi: A \times \pi_1(X) \to A$. *If* (X, Y) *is* $(n-1)$*-connected for* $n \geq 2$, *then there is an isomorphism*

$$h: H_n(X, Y; A_\phi) \approx A \otimes_{\pi_1(Y)} \pi_n(X, Y).$$

Proof. Let $\widetilde{X}$ be the universal covering of X with covering projection $p: \widetilde{X} \to X$. Then $\widetilde{Y} = p^{-1}(Y)$ is the universal covering of Y, and $\pi_1(Y)$ operates properly on the pair $(\widetilde{X}, \widetilde{Y})$. Set $\pi = \pi_1(Y)$. Since the singular complex $\Delta(\widetilde{X})/\Delta(\widetilde{Y})$ is π-free, Theorem 8.4 in Chapter XVI of [1] is valid. It follows that there is a convergent spectral sequence

$$H_p(\pi, H_q(\widetilde{X}, \widetilde{Y}; A)) \underset{p}{\Rightarrow} H_n(X, Y, A_\phi).$$

Because (X, Y) is $(n-1)$-connected, $E^2_{p,q} = 0$ if $p < 0$ or $q < n$. The exact sequence of Theorem 5.12a in Chapter XV of [1] is reduced to an isomorphism $E^2_{0,n} \approx H_n$. This with the universal-coefficient formula and classical Hurewicz theorem yields the proposition as follows,

$$H_n(X, Y; A_\phi) \approx H_0(\pi, H_n(\widetilde{X}, \widetilde{Y}; A)) \approx [H_n(\widetilde{X}, \widetilde{Y}; A)]_\pi$$
$$\approx A \otimes_\pi H_n(\widetilde{X}, \widetilde{Y}) \approx A \otimes_\pi \pi_n(X, Y). \quad \text{Q.E.D.}$$

2.2. Thus we have reduced the proof of Theorem 0.1 to the computation of the local system $Z_{\Psi(f)}$ and the relative homotopy group $\pi_{n-k+1}(\Lambda_f \times_2 S^\infty, M \times P^\infty)$ with the action of $\pi_1(M \times P^\infty)$ on it.

The local system $Z_{\Psi(f)}$ was determined by Proposition 2.3 of [12]. Its restriction on $\pi_1(M \times P^\infty) = \pi_1(M) \times T_2$ is characterized by a homomorphism $\Psi: \pi_1(M \times P^\infty) \to \operatorname{Aut} Z$ such that

$$\Psi(a, 1) = \Psi(e)(a, a; 1) = (-1)^{f_\pi(a)},$$
$$\Psi(a, m) = \Psi(e)(a, a; m) = (-1)^k \cdot (-1)^{f_\pi(a)}$$

for $a \in \pi_1(M)$, where $T_2 = \{1, m\}$ is the multiplicative group $\pi_1(P^\infty)$ of two elements.

Now denote by $*$ the basepoint of M, s the basepoint of S^∞. We take $(*, s)$ to the basepoint of $\Lambda_f \times S^\infty$. Let $p: \Lambda_f \times S^\infty \to \Lambda_f \times_2 S^\infty$ be the quotient map and let $p_1\colon \Lambda_f \times S^\infty \to \Lambda_f = P(V; M, M)$ be the projection to the first factor. They induce isomorphisms of relative homotopy groups

$$\pi_{n-k+1}(\Lambda_f \times_2 S^\infty, M \times P^\infty) \xleftarrow{p_*} \pi_{n-k+1}(\Lambda_f \times S^\infty, M \times S^\infty) \xrightarrow{p_{1*}} \pi_{n-k+1}(\Lambda_f, M) \xrightarrow{A} \pi_{n-k+2}(V, M).$$

where A is given in Proposition 1.2. It is clear that the composition $A \circ p_{1*} \circ p_*^{-1}$ is commutative with the operations of $\pi_1(M)$.

Now we consider the action h_m of

$$m \in \pi_1(P^\infty) \subset \pi_1(M \times P^\infty)$$

on $\pi_{n-k+1}(\Lambda_f \times_2 S^\infty, M \times P^\infty)$. Represent m by a loop $(c_*, \rho)\colon I \to M \times P^\infty$ where c_* is the constant loop of M on $*$, ρ is a loop of P^∞ based at $[s]$ whose lifting to S^∞ is a path $\tilde\rho$ from s to $-s$. Denote by $\tilde m$ the path class of $(c_*, \tilde\rho)$ in $M \times S^\infty$. Lifting h_m to the double covering p, we get $h_{\tilde m} \cdot T_*$ which is fitted with the following commutative diagram:

$$\begin{array}{ccccccc}
\pi_{n-k+1}(\Lambda_f \times_2 S^\infty, M \times P^\infty) & \longleftarrow & \pi_{n-k+1}(\Lambda_f \times S^\infty, M \times S^\infty) & \longrightarrow & \pi_{n-k+1}(\Lambda_f, M) & \xrightarrow{A} & \pi_{n-k+2}(V, M) \\
\downarrow{\scriptstyle h_m} & & \downarrow{\scriptstyle h_{\tilde m} \circ T_*} & & \downarrow{\scriptstyle T_*} & & \downarrow{\scriptstyle -1} \\
\pi_{n-k+1}(\Lambda_f \times_2 S^\infty, M \times P^\infty) & \longleftarrow & \pi_{n-k+1}(\Lambda_f \times S^\infty, M \times S^\infty) & \longrightarrow & \pi_{n-k+1}(\Lambda_f, M) & \xrightarrow{A} & \pi_{n-k+2}(V, M)
\end{array}$$

where $T\colon \Lambda_f \times S^\infty \to \Lambda_f \times S^\infty$ and $\Lambda_f \to \Lambda_f$ are involutions defined in §1. From 1.3, 2.1, and 2.2, it follows that

2.3. **Proposition.** *Let $f\colon M^n \to V^{n+k}$ be $(n-k+1)$-connected for $n \le 2k-4$ and $k < n$. Then $[M \subset V]_f$ is the quotient group of $\pi_{n-k+2}(V, M)$ by the subgroup $\{(-1)^{f(a)}x - h_a(x)\colon x \in \pi_{n-k+2}(V, M), a \in \pi_1(M)\}$ if k is odd and the tensor product of Z_2 with the quotient group of $\pi_{n-k+2}(V, M)$ by the subgroup $\{x - h_a(x)\colon x \in \pi_{n-k+2}(V, M), a \in \pi_1(M)\}$ if k is even where $h_a\colon \pi_{n-k+2}(V, M) \to \pi_{n-k+2}(V, M)$ is the action of $a \in \pi_1(M)$.*

This with Proposition 2.1 yields our Theorem 0.1.

3. Proof of Theorem 0.2

Since Proposition 2.1 is not valid for $n = 1$, we are obliged to seek another way to calculate $H_1(X, Y; Z_\phi)$. This problem can be converted into computation of twisted abstract homology groups by the exact sequence

$$H_1(Y; Z_{\phi|Y}) \to H_1(X; Z_\phi) \to H_1(X, Y; Z_\phi) \to H_0(Y; Z_{\phi|Y}) \to H_0(X; Z_\phi).$$

The following theorem offers such a calculation method.

3.1. Theorem. *Let X be a path-connected topological space. Suppose that Z_ϕ is a local system of integers on X characterized by a homomorphism $\phi\colon \pi_1(X) \to \operatorname{Aut} Z$. Then there is an isomorphism*

$$H_1(X; Z_\phi) \cong \pi_1^+/([\pi_1^+, \pi_1^+]\cdot[\pi_1^-]^2),$$

where $\pi_1^+ = \ker\phi$, $\pi_1^- = \pi_1(X)\backslash\pi_1^+$, $[\pi_1^+, \pi_1^+]$ is the commutative group of π_1^+, and $[\pi_1^-]^2$ is the normal subgroup of π_1^+ generated by the elements x^2 for $x \in \pi_1^-$.

Proof. Let $C_*^0(X; Z_\phi)$ denote the chain complex of X with the local coefficients Z_ϕ generated by singular simplexes $u\colon \Delta^n \to X$, all of whose vertices are at the basepoint of X. We define $H_n(X; Z_\phi)$ as the nth homology group of $C_*^0(X; Z_\phi)$. We shall construct a homomorphism $\Psi\colon H_1(X; Z_\phi) \to \pi_1^+/([\pi_1^+, \pi_1^+]\cdot[\pi_1^-]^2)$ and its inverse. For convenience let $[\sigma]$ denote the homotopy class of a loop σ, and let $\overline{[\sigma]}$ be the equivalent class of $[\sigma] \bmod [\pi_1^+, \pi_1^+]\cdot[\pi_1^-]^2$.

Choose a singular 1-simplex $\rho \in C_1^0(X; Z_\phi)$ such that $[\rho] \in \pi_1^-$ if it exists. Then $Z_1(X; Z_\phi)$ is a free abelian group generated by singular simplexes σ such that $[\sigma] \in \pi_1^+$ and the differences $\sigma - \rho$ such that $[\sigma] \in \pi_1^-$. Thus there is a homomorphism $\psi\colon Z_1(X; Z_\phi) \to \pi_1^+/([\pi_1^-, \pi_1^+]\cdot 0u[\pi_1^-]^2)$ defined by $\psi(\sigma) = \overline{[\sigma]}$ for $[\sigma] \in \pi_1^+$, and $\psi(\sigma - \rho) = \overline{[\sigma * \rho]}$ for $[\sigma] \in \pi_1^-$. Now we prove ψ annihilates the subgroup $\operatorname{Im}(\partial\colon C_2^0(X; Z_\phi) \to C_1^0(X; Z_\phi)) \subset Z_1(X; Z_\phi)$.

Notice that for any 2-simplex $u\colon \Delta^2 \to X$ in $C_2^0(X; Z_\phi)$ we have $\partial u = \phi([u^{(2)}])u^{(0)} - u^{(1)} + u^{(2)}$ where $u^{(i)}$ is the opposite face of the ith vertex of u. Certainly their homotopy classes satisfy the relation $[u^{(1)}] = [u^{(2)}]\cdot[u^{(0)}]$.

Now if $[u^{(i)}] \in \pi_1^+$ for $i = 0, 1, 2$, then

$$\psi(\partial u) = \overline{[u^{(0)}]\cdot[u^{(1)}]^{-1}\cdot[u^{(2)}]} = \bar{e}.$$

If $[u^{(2)}] \in \pi_1^+$ and $[u^{(0)}] \in \pi_1^-$, then $[u^{(1)}] \in \pi_1^-$. In this case, we replace ∂u by $(u^{(0)} - \rho) - (u^{(1)} - \rho) + u^{(2)}$. It follows that

$$\psi(\partial u) = \overline{[u^{(0)} * \rho]\cdot[u^{(1)} * \rho]^{-1}\cdot[u^{(2)}]} = \bar{e}.$$

If $[u^{(2)}]$ and $[u^{(0)}] \in \pi_1^-$, then $[u^{(1)}] \in \pi_1^+$ and $\partial u = (u^{(2)} - \rho) - (u^{(0)} - \rho) - u^{(1)}$. Thus

$$\begin{aligned}\psi(\partial u) &= \overline{[u^{(2)} * \rho]\cdot[u^{(0)} * \rho]^{-1}\cdot[u^{(1)}]^{-1}}\\ &= \overline{[u^{(2)}][u^{(0)}]^{-1}[u^{(0)}]^{-1}\cdot[u^{(2)}]^{-1}}\\ &= \overline{([u^{(2)}]\cdot[u^{(0)}]^{-1}\cdot[u^{(2)}]^{-1})^2} = \bar{e}.\end{aligned}$$

If $[u^{(2)}]$ and $[u^{(1)}] \in \pi_1^-$, then $[u^{(0)}] \in \pi_1^+$ and $\partial u = -u^{(0)} + (u^{(2)} - \rho) - (u^{(1)} - \rho)$. Therefore

$$\begin{aligned}\psi(\partial u) &= \overline{[u^{(0)}]^{-1}\cdot[u^{(2)} * \rho]\cdot[u^{(1)} * \rho]^{-1}}\\ &= \overline{[u^{(0)}]^{-1}[u^{(2)}][u^{(0)}]^{-1}\cdot[u^{(2)}]^{-1}}\\ &= \overline{([u^{(0)}]^{-1}\cdot[u^{(2)}])^2\cdot([u^{(2)}]^{-1})^2} = \bar{e}.\end{aligned}$$

Summing up the above discussion, we obtain a homomorphism $\Psi\colon H_1(X; Z_\phi) \to \pi_1^+/([\pi_1^+, \pi_1^+]\cdot[\pi_1^-]^2)$ as the quotient of ψ. In the rest of the proof, we define a homomorphism $\Phi\colon \pi_1^+/([\pi_1^+, \pi_1^+]\cdot[\pi_1^-]^2) \to H_1(X; Z_\phi)$ and show that Φ is an inverse of Ψ.

Let σ be a loop such that $[\sigma] \in \pi_1^+$. Then σ is a cycle in $Z_1(X; Z_\phi)$ as well and we denote by $\{\sigma\}$ its homology class in $H_1(X; Z_\phi)$. Hence there is a homomorphism $\varphi: \pi_1^+ \to H_1(X; Z_\phi)$ given by $\varphi([\sigma]) = \{\sigma\}$. It is easy to verify that φ annihilates the commutator group of π_1^+, and it is sufficient to prove $\varphi(x^2) = 0$ for $x \in \pi_1^-$. In fact, let $u: \Delta^2 \to X$ be a 2-simplex such that $u^{(2)} = u^{(0)}$ representing x; then $[u^{(1)}] = x^2$, and $\partial u = -u^{(0)} - u^{(1)} + u^{(2)} = -u^{(1)}$. It follows that $0 = \{u^{(1)}\} = \varphi(x^2)$. Taking the quotient of φ, we obtain a homomorphism

$$\Phi: \pi_1^+/([\pi_1^+, \pi_1^+] \cdot [\pi_1^-]^2) \to H_1(X; Z_\phi).$$

It is clear that $\Psi \cdot \Phi = I$. Now let us consider $\Phi \cdot \Psi$. If σ is a 1-simplex in $C_1^0(X; Z_\phi)$ such that $[\sigma] \in \pi_1^+$, then $\Phi \cdot \Psi(\{\sigma\}) = \Phi(\overline{[\sigma]}) = \{\sigma\}$. If σ is a 1-simplex in $C_1^0(X; Z_\phi)$ such that $[\sigma] \in \pi_1^-$, then we can construct a 2-simplex $u: \Delta^2 \to X$ such that $\sigma^{(2)} = \sigma$, $u^{(0)} = \rho$, and $u^{(1)} = \sigma * \rho$. Hence $\partial u = -\rho - \sigma * \rho + \sigma$ and

$$\Phi \cdot \Psi(\{\sigma - \rho\}) = \Phi(\overline{[\sigma * \rho]}) = \{\sigma * \rho\} = \{\sigma - \rho\}.$$

Therefore we have $\Phi \cdot \Psi = I$ and the theorem. Q.E.D.

3.2. **Corollary.** *In addition to the hypotheses of Theorem* 3.1, *suppose that* $i: Y \subset X$ *is a path connected subspace. Then*

$$H_1(X, Y; Z_\phi) \cong \begin{cases} \dfrac{\pi_1^+(X)}{[\pi_1^+(X), \pi_1^+(X)] \cdot [\pi_1^-(X)]^2 \cdot i_\pi(\pi_1^+(Y))} + Z & \text{if } \phi \neq 1 \text{ and } \phi \cdot i_\pi = 1, \\ \dfrac{\pi_1^+(X)}{[\pi_1^+(X), \pi_1^+(X)] \cdot [\pi_1^-(X)]^2 \cdot i_\pi(\pi_1^+(Y))} & \text{otherwise.} \end{cases}$$

Now we start on our proof of 0.1.

3.3. **Lemma.**

$$[(x, e, 1), (e, a, 1)] = (x \cdot h_a(x^{-1}), e, 1),$$
$$[(x, e, 1), (e, a, m)] = (x \cdot h_a(x), a \cdot \partial x^{-1} \cdot a^{-1}, 1),$$
$$(x, a, 1)^2 = (x \cdot h_a(x), a^2, 1),$$
$$(x, a, m)^2 = (x \cdot h_a(x^{-1}), a \cdot \partial x \cdot a, 1).$$

Proof. It can be directly verified. Q.E.D.

For convenience, let $\pi_2^{(+)}(V, M)$ denote $\partial^{-1}(\pi_1^{(+)}(M))$ for $\partial: \pi_2(V, M) \to \pi_1(M)$ and let $\pi_1^{(+)}(M)$ denote $f_\pi^{-1}(\pi_1^{(+)}(V))$ for $f_\pi: \pi_1(M) \to \pi_1(V)$. Set $\pi_2^{(-)}(V, M) = \pi_2(V, M) \backslash \pi_2^{(+)}(V, M)$, $\pi_1^{(-)}(M) = \pi_1(M) \backslash \pi_1^{(+)}(M)$. Generally $\pi_1^{(\pm)}(M)$ are different from $\pi_1^{\pm}(M)$.

From the assumption that $\ker f_\pi \subseteq \pi_1^+(M)$, it follows that $\pi_2^{(-)}(V, M) = \phi$. By Proposition 1.3, the orientability of $\Psi(f)$ is completely determined by its restriction on $M \times P^\infty$.

If n is even, then $\pi_1^+(M \times P^\infty) = \pi_1^{(+)}(M) \times T_2$ and $\pi_1^+(\Lambda_f \times_2 S^\infty) = \pi_2(V, M) \cdot \pi_1^+(M \times P^\infty)$. In this case, $[\pi_1^+(\Lambda_f \times_2 S^\infty), \pi_1^+(\Lambda_f \times_2 S^\infty)]$ is generated by the commutators of subgroups $\pi_2(V, M)$, $\pi_1^{(+)}(M) \times T_2$ and the commutators between them:

$$[(x, e, 1), (e, a, 1)] \quad \text{and} \quad [(x, e, 1), (e, a, m)] \quad \text{for } x \in \pi_2(V, M), a \in \pi_1^{(+)}(M).$$

$[\pi_1^-(\Lambda_f \times_2 S^\infty)]^2$ is generated by $(x, a, 1)^2$ and $(x, a, m)^2$ for $x \in \pi_2(V, M)$, $a \in \pi_1^{(-)}(M)$. It follows from Lemma 3.3 that $H_1(\Lambda_f \times_2 S^\infty, M \times P^\infty; Z_{\Psi(f)})$ is isomorphic to the quotient group of $\pi_2(V, M)$ by the normal subgroup generated by the commutators of $\pi_2(V, M)$ and the following elements:

$$x \cdot h_a(x^{-1}), x \cdot h_a(x) \quad \text{for } x \in \pi_2(V, M), a \in \pi_1^{(+)}(M),$$
$$x \cdot h_a(x), x \cdot h_a(x^{-1}) \quad \text{for } x \in \pi_2(V, M), a \in \pi_1^{(-1)}(M).$$

Since $x \cdot h_a(x) = x^2 \cdot (x^{-1} h_a(x))$, we obtain that

$$H_1(\Lambda_f \times_2 S^\infty, M \times P^\infty; Z_{\Psi(f)}) \cong \frac{\pi_2(V, M)}{\langle x \cdot h_a(x^{-1}) \colon x \in \pi_2(V, M), a \in \pi_1(M)\rangle} \otimes Z_2 = H_2(V, M; Z_2).$$

If n is odd, then

$$\pi_1^+(M \times P^\infty) = \{(e, a, 1) \colon a \in \pi_1^{(+)}(M)\} \cup \{(e, a, m) \colon a \in \pi_1^{(-)}(M)\}$$

and

$$\pi_1^+(\Lambda_f \times_2 S^\infty) = \pi_2(V, M) \cdot \pi_1^+(M \times P^\infty).$$

Hence $[\pi_1^+(\Lambda_f \times_2 S^\infty), \pi_1^+(\Lambda_f \times_2 S^\infty)]$ is generated by the commutators of $\pi_2(V, M)$, $\pi_1^+(M \times P^\infty)$ and the commutators between them:

$$[(x, e, 1), (e, a, 1)] \quad \text{for } x \in \pi_2(V, M), a \in \pi_1^{(+)}(M),$$
$$[(x, e, 1), (e, a, m)] \quad \text{for } x \in \pi_2(V, M), a \in \pi_1^{(-)}(M).$$

$[\pi_1^-(\Lambda_f \times_2 S^\infty)]^2$ is generated by

$$(x, a, 1)^2 \quad \text{for } x \in \pi_2(V, M), a \in \pi_1^{(-)}(M),$$
$$(x, a, m)^2 \quad \text{for } x \in \pi_2(V, M), a \in \pi_1^{(+)}(M).$$

By Lemma 3.3, $H_1(\Lambda_f \times_2 S^\infty, M \times P^\infty; Z_{\Psi(f)})$ is isomorphic to the quotient group of $\pi_2(V, M)$ by the normal subgroup generated by the commutators of $\pi_2(V, M)$, the elements $x \cdot h_a(x^{-1})$ for $x \in \pi_2(V, M)$, $a \in \pi_1^{(+)}(M)$ and the elements $x \cdot h_a(x)$ for $x \in \pi_2(V, M)$, $a \in \pi_1^{(-)}(M)$.

The following lemma allows us to complete the proof.

3.4. **Lemma.** *Let (X, Y) be a 1-connected couple of path-connected spaces and let Z_ϕ be a local system on X characterized by a homomorphism $\phi \colon \pi_1(X) \to \operatorname{Aut} Z$. Then $H_2(X, Y; Z_\phi)$ is the quotient group of $\pi_2(X, Y)$ by the normal subgroup generated by the elements*

$$x^{-1} \cdot h_a(x) \quad \textit{for } x \in \pi_2(X, Y), a \in \pi_1^{(+)}(Y),$$
$$x \cdot h_a(x) \quad \textit{for } x \in \pi_2(X, Y), a \in \pi_1^{(-)}(Y),$$

where $\pi_1^{(+)}(Y)$ *is the kernel of composition* $\pi_1(Y) \xrightarrow{i_\pi} \pi_1(X) \xrightarrow{\phi} Z_2$, $\pi_1^{(-)}(Y) = \pi_1(Y)\backslash\pi_1^{(+)}(Y)$.

Proof. Let $\hat{\pi}_2(X, Y)$ denote the quotient group of $\pi_2(X, Y)$ presented in this lemma. Because (X, Y) is 1-connected, the singular chain complex $C_*(X, Y; Z_\phi)$ is chain homotopic to the normal singular chain complex $C_*^{(1)}(X, Y; Z_\phi)$ which is generated by singular simplexes $\sigma: \Delta^q \to X$ having the property that σ maps each vertex of Δ^q to the basepoint of $Y \subset X$ and maps the 1-dimensional skeleton $(\Delta^q)^1$ to Y. Each singular simplex $\sigma: (\Delta^2, (\Delta^2)^1, (\Delta^2)^0) \to (X, Y, *)$ determines an element $[\sigma] \in \hat{\pi}_2(X, Y)$. Since $\hat{\pi}_2(X, Y)$ is abelian, this defines a homomorphism $\psi: C_2^{(1)}(X, Y; Z_\phi) \to \hat{\pi}_2(X, Y)$.

We show that Ψ annihilates $\partial C_3^{(1)}(X, Y; Z_\phi)$. Let $\sigma \in C_3^{(1)}(X, Y; Z_\phi)$ be a simplex $\sigma: \Delta^3, (\Delta^3)^1, (\Delta^3)^0 \to X, Y, *$. Then

$$\partial\sigma = \phi(w_\sigma)\sigma^{(0)} + \sum_{0<i\leq 3}(-1)^i\sigma^{(i)},$$

where $w_\sigma: I \to Y$ is the restriction of σ on the edge $v_0v_1 \subset \Delta^3$. If $\phi(w_\sigma) = 1$, then

$$\begin{aligned}\psi\partial(\sigma) &= \psi(\sigma^{(0)}) \cdot \psi(\sigma^{(2)})[\psi(\sigma^{(1)})]^{-1}[\psi(\sigma^{(3)})]^{-1}\\ &= \psi(h_{w_\sigma}(\sigma^{(0)}))[\psi(\sigma^{(0)})]^{-1}\psi(\sigma^{(0)})\psi(\sigma^{(2)})[\psi(\sigma^{(1)})]^{-1}[\psi(\sigma^{(3)}]^{-1}\\ &= \psi(h_{w_\sigma}(\sigma^{(0)}))\psi(\sigma^{(2)})[\psi(\sigma^{(1)}]^{-1}[\psi(\sigma^{(3)})]^{-1}.\end{aligned}$$

The homotopy addition theorem asserts that $\psi\partial(\sigma) = 0$. Similarly, if $\phi(w_\sigma) = -1$, then

$$\begin{aligned}\psi\partial(\sigma) &= [\psi(\sigma^{(0)})]^{-1} \cdot \psi(\sigma^{(2)})[\psi(\sigma^{(1)})]^{-1}[\psi(\sigma^{(3)})]^{-1}\\ &= \psi(h_{w_\sigma}(\sigma^{(0)}))\psi(\sigma^{(0)})[\psi(\sigma^{(0)})]^{-1}\psi(\sigma^{(2)})[\psi(\sigma^{(1)})]^{-1}[\psi(\sigma^{(3)})]^{-1} = 0.\end{aligned}$$

Therefore ψ defines a homomorphism $\Psi: H_2(X, Y; Z_\pi) \to \hat{\pi}_2(X, Y)$. Conversely, consider each map $\alpha: \Delta^2, \dot{\Delta}^2, v_0 \to X, Y, *$ as a simplex in $C_2(X, Y; Z_\phi)$. In fact, it is a cycle. It follows that there is a map

$$h: (X, Y, *)^{(\Delta^2, \dot{\Delta}^2, v_0)} \to H_2(X, Y; Z_\phi).$$

Now we observe the effect of homotopy. Let $F: \Delta^2 \times I, \dot{\Delta}^2 \times I \to X, Y$ be a homotopy from α to α'. Then their homotopy classes satisfy $[\alpha'] = h_{[w^{-1}]}([\alpha])$ where $w = F|_{v_0 \times I}$ is a loop of Y based at $*$. Set $v_i' = v_i \times 0$, $v_i'' = v_i \times 1$ for the vertices $v_i \in \Delta^2$. We triangulate $\Delta^2 \times I$ by 3-simplexes $\Delta_1^3, \Delta_2^3, \Delta_3^3$ and their faces where $\Delta_1^3 = v_0'v_0''v_1''v_2''$, $\Delta_2^3 = v_0'v_1'v_1''v_2''$, $\Delta_3^3 = v_0'v_1'v_2'v_2''$. Let F_i denote the 3-simplexes $F|_{\Delta_i^3} \in C_3(X, Y; Z_\phi)$. A direct calculation shows that $\partial(F_1 - F_2 + F_3) = \phi([w])\alpha' - \alpha$. It follows that their homology classes satisfy $\{\alpha\} = \phi([w])\{\alpha'\}$. Taking the quotient of h under the homotopy, we get a homomorphism $H: \hat{\pi}_2(X, Y) \to H_2(X, Y; Z_\phi)$. One can directly verify that H is the inverse of Ψ. Q.E.D.

4. Proof of Theorem 0.3

Since V is orientable and $\ker f_\pi \not\subseteq \pi_1^+(M)$, we have $\pi_1^{(-)}(M) = \phi$ and $\pi_2^{(-)}(V, M) \neq \phi$.

If n is even, then $\pi_1^+(M \times P^\infty) = \pi_1(M \times P^\infty)$,

$$\pi_1^+(\Lambda_f \times_2 S^\infty) = (\pi_2^{(+)}(V, M) \times_h \pi_1(M)) \times_\phi T_2.$$

It is clear that $[\pi_1^+(\Lambda_f \times_2 S^\infty), \pi_1^+(\Lambda_f \times_2 S^\infty)]$ is generated by the commutators of subgroups $\pi_2^{(+)}(V, M)$, $\pi_1(M) \times T_2$ and the commutators between them:

$$[(x, e, 1), (e, a, 1)] \quad \text{and} \quad [(x, e, 1), (e, a, m)] \qquad \text{for } x \in \pi_2^{(+)}(V, M),\ a \in \pi_1(M).$$

$[\pi_1^-(\Lambda_f \times_2 S^\infty)]^2$ is generated by $(x, a, 1)^2$ and $(x, a, m)^2$ for $x \in \pi_2^{(-)}(V, M)$, $a \in \pi_1(M)$. From Lemma 3.3, $H_1(\Lambda_f \times_2 S^\infty, M \times P^\infty; Z_{\Psi(f)})$ is isomorphic to the direct sum of Z with the quotient group of $\pi_2^{(+)}(V, M)$ by the normal subgroup generated by the commutators of $\pi_2^{(+)}(V, M)$ and the following elements:

$$x \cdot h_a(x^{-1}),\ x \cdot h_a(x) \quad \text{for } x \in \pi_2^{(+)}(V, M),\ a \in \pi_1(M),$$
$$x \cdot h_a(x),\ x \cdot h_a(x^{-1}) \quad \text{for } x \in \pi_2^{(-)}(V, M),\ a \in \pi_1(M).$$

Notice that $x \cdot h_a(x) = x^2 \cdot (x^{-1} h_a(x))$. By the classical Hurewicz theorem, we obtain

$$H_1(\Lambda_f \times_2 S^\infty, M \times P^\infty; Z_{\Psi(f)}) \cong Z + H_2^{(+)}(V, M; Z_2),$$

where $H_2^{(+)}(V, M; Z_2)$ is the kernel of the composition $H_2(V, M; Z_2) \xrightarrow{\partial} H_1(M; Z_2) \xrightarrow{w_1(M)} Z_2$.

Now we discuss the other case. A direct computation shows that

4.1. Lemma.

$$[(x, e, 1), (\overline{x}, e, m)] = (x^2, \partial x^{-1}, 1),$$
$$[(e, a, 1), (\overline{x}, e, m)] = (h_a(\overline{x}) \cdot \overline{x}^{-1}, e, 1).$$

If n is odd, then $\pi_1^+(M \times P^\infty) = \pi_1(M)$, $\pi_1^+(\Lambda_f \times_2 S^\infty) = A \cup B$ where $A = \{(x, a, 1) : x \in \pi_2^{(+)}(V, M),\ a \in \pi_1(M)\}$, $B = \{(x, a, m) : x \in \pi_2^{(-)}(V, M),\ a \in \pi_1(M)\}$. Choose an element $\overline{x} \in \pi_2^{(-)}(V, M)$. Certainly, each element of B can be uniquely decomposed as a product $(x, a, 1) \cdot (\overline{x}, e, m)$ where $(x, a, 1) \in A$. It follows that $[\pi_1^+(\Lambda_f \times_2 S^\infty), \pi_1^+(\Lambda_f \times_2 S^\infty)]$ is generated by the commutators of subgroups $\pi_2^{(+)}(V, M)$, $\pi_1(M)$, the commutators $[(x, e, 1), (e, a, 1)]$, $[(x, e, 1), (\overline{x}, e, m)]$ and $[(e, a, 1), (\overline{x}, e, m)]$ for $x \in \pi_2^{(+)}(V, M)$ and $a \in \pi_1(M)$. $[\pi_1^-(\Lambda_f \times_2 S^\infty)]^2$ is generated by the elements

$$(x, a, 1)^2, \quad for\ x \in \pi_2^{(-1)}(V, M),\ a \in \pi_1(M),$$
$$(x, a, m)^2 \quad for\ x \in \pi_2^{(+)}(V, M),\ a \in \pi_1(M).$$

Because we have $(\overline{x}, e, m)^2 = (e, \partial \overline{x}, 1)$, the image of $(\overline{x}, e, m)$ in $H_1(\Lambda_f \times_2 S^\infty, M \times P^\infty; Z_{\Psi(f)})$ is of order 2. By using Lemmas 3.3 *and* 4.1*, a discussion similar to the case that n is even shows that*

$$H_1(\Lambda_f \times_2 S^\infty, M \times P^\infty; Z_{\Psi(f)}) \approx H_2^{(+)}(V, M; Z_2) + Z_2 \approx H_2(V, M; Z_2).$$

5. Proof of Theorem 0.4

First of all, one can verify

5.1. Lemma.

$$[(x, e, 1), (\overline{x}, \overline{a}, 1)] = (x \cdot \overline{x} \cdot h_{\overline{a}}(x^{-1}) \cdot \overline{x}^{-1}, e, 1),$$
$$[(e, a, 1), (\overline{x}, \overline{a}, 1)] = (h_a(\overline{x}) \cdot h_{[a, \overline{a}]}(\overline{x}^{-1}), [a, \overline{a}], 1),$$
$$[(e, e, m), (\overline{x}, \overline{a}, 1)] = (\overline{x}^{-2}, \partial\overline{x}, 1),$$
$$[(e, \overline{a}, m), (\overline{x}, e, m)] = (\overline{x}^{-1} \cdot h_{\overline{a}}(\overline{x}^{-1}), \overline{a} \cdot \partial\overline{x} \cdot \overline{a}^{-1}).$$

By the assumption that $\ker f_\pi \not\subset \pi_1^+(M)$ and $w_1(V) \neq 0$, we obtain that $\pi_1^{(-)}(V, M) \neq \phi$ and $\pi_1^{(-)}(M) \neq \phi$ and take an element $\overline{x} \in \pi_2^{(-)}(V, M)$ and an element $\overline{a} \in \pi_1^{(-)}(M)$.

If n is even, then $\pi_1^+(M \times P^\infty) = \pi_1^{(+)}(M) \times T_2$ and $\pi_1^+(\Lambda_f \times_2 S^\infty) = A \cup B$ where

$$A = \pi_1^{(+)}(V, M) \cdot \pi_1^+(M \times P^\infty) = (\pi_2^{(+)}(V, M) \times_h \pi_1^{(+)}(M)) \times_\phi T_2,$$
$$B = \pi_2^{(-)}(V, M) \cdot \pi_1^-(M \times P^\infty).$$

It is evident that the elements of B can be uniquely decomposed as products $\xi \cdot (\overline{x}, \overline{a}, 1)$ for $\xi \in A$. Thus $[\pi_1^+(\Lambda_f \times_2 S^\infty), \pi_1^+(\Lambda_f \times_2 S^\infty)]$ is generated by the commutators of subgroup $\pi_2^{(+)}(V, M)$, $\pi_1^+(M \times P^\infty)$, and the following commutators:

$$[(x, e, 1), (e, a, 1)], \quad [(x, e, 1), (e, a, m)], \quad [(x, e, 1), (\overline{x}, \overline{a}, 1)],$$

$$[(e, a, 1), (\overline{x}, \overline{a}, 1)], \quad [(e, e, m), (\overline{x}, \overline{a}, 1)]$$

for $x \in \pi_2^{(+)}(V, M)$, $a \in \pi_1^{(+)}(M)$. On the other hand, $[\pi_1^-(\Lambda_f \times_2 S^\infty)]^2$ is generated by $(x', a, 1)^2$, $(x', a, m)^2$, $(x, a', 1)^2$, and $(x, a', m)^2$ for $x \in \pi_2^{(+)}(V, M)$, $x' \in \pi_2^{(-)}(V, M)$, $a \in \pi_1^{(+)}(M)$, and $a' \in \pi_1^{(-)}(M)$. By Lemmas 3.3 and 5.1, $H_1(\Lambda_f \times_2 S^\infty, M \times P^\infty; Z_{\Psi(f)})$ has a subgroup isomorphic to the quotient group of $\pi_2^{(+)}(V, M)$ by its normal subgroup H generated by its commutators and the following elements: $x^{\pm 1}h_a(x)$, $x'^{\pm 1}h_a(x')$, and $x^{\pm 1}h_{a'}(x)$ for $x \in \pi_2^{(+)}(V, M)$, $x' \in \pi_2^{(-)}(V, M)$, $a \in \pi_1^{(+)}(M)$, and $a' \in \pi_1^{(-)}(M)$. Because the elements $x'^{\pm 1}h_{a'}(x')$ are not necessarily in H for $x' \in \pi_2^{(-)}(V, M)$ and $a' \in \pi_1^{(-)}(M)$, in general the obtained quotient group $\pi_2^{(+)}(V, M)/H$ is not $H_2^{(+)}(V, M; Z_2)$.

Since $\pi_2^{(+)}(V, M) = \pi_2^{(+)}(\overline{V}, \overline{M})$ and $\pi_1(\overline{M}) = \pi_1^{(+)}(M)$, the quotient group of $\pi_2^{(+)}(V, M)$ by its normal subgroup, H' generated by the elements $x^{\pm 1}h_a(x)$ for $x \in \pi_2(V, M)$, $a \in \pi_1^{(+)}(M)$, is just $H_2^{(+)}(\overline{V}, \overline{M}; Z_2)$ by the classical Hurewicz theorem. In this quotient group, the images of $x^{\pm} \cdot h_{a'}(x)$ generate the subgroup $\langle x + T_*(x) : x \in H_2^{(+)}(\overline{V}, \overline{M}; Z_2)\rangle \subset H_2^{(+)}(\overline{V}, \overline{M}; Z_2)$, where $x \in \pi_2^{(+)}(V, M)$, $a' \in \pi_1^{(-)}(M)$, and $T_* : H_2^{(+)}(\overline{V}, \overline{M}; Z_2) \to H_2^{(+)}(\overline{V}, \overline{M}; Z_2)$ is induced by the covering involution $T : \overline{V}, \overline{M} \to \overline{V}, \overline{M}$. It follows that

$\pi_2^{(+)}(V, M)/H \cong H_2^{(+)}(\overline{V}, \overline{M}; Z_2)/\langle x + T_*(x): x \in H_2^{(+)}(\overline{V}, \overline{M}; Z_2)\rangle$ and the first part of Theorem 0.4 is proved.

If n is odd, then

$$\pi_1^+(M \times P^\infty) = \{(e, a, 1): a \in \pi_1^{(+)}(M)\} \cup \{(e, a', m): a' \in \pi_1^{(-)}(M)\}$$

and

$$\pi_1^+(\Lambda_f \times_2 S^\infty) = \pi_2^{(+)}(V, M) \cdot \pi_1^+(M \times P^\infty) \cup \pi_2^{(-)}(V, M) \cdot \pi_1^-(M \times P^\infty).$$

Since the elements of $\pi_2^{(-)}(V, M) \cdot \pi_1^-(M \times P^\infty)$ can be decomposed as products $\xi \cdot (\overline{x}, e, m)$ for $\xi \in \{\pi_2^{(+)}(V, M) \cdot \pi_1^+(M \times P^\infty)\}$, the commutator group $[\pi_1^+(\Lambda_f \times_2 S^\infty), \pi_1^+(\Lambda_f \times_2 S^\infty)]$ is generated by the commutators of subgroups $\pi_2^{(+)}(V, M)$, $\pi_1^+(M \times P^\infty)$, and the following elements:

$$[(x, e, 1), (e, a, 1)], \quad [(x, e, 1), (e, a', m)],$$
$$[(x, e, 1), (\overline{x}, e, m)], \quad [(e, a, 1), (\overline{x}, e, m)], \quad [(e, \overline{a}, m), (\overline{x}, e, m)]$$

for $x \in \pi_2^{(+)}(V, M)$, $a \in \pi_1^{(+)}(M)$, $a' \in \pi_1^{(-)}(M)$. $[\pi_1^-(\Lambda_f \times_2 S^\infty)]^2$ is generated by the elements $(x, a', 1)^2$, $(x, a, m)^2$, $(x', a, 1)^2$ for $x \in \pi_2^{(+)}(V, M)$, $x' \in \pi_2^{(-)}(V, M)$ and $a \in \pi_1^{(+)}(M)$, $a' \in \pi_1^{(-)}(M)$. From Lemmas 3.3, 4.1, and 5.1, it follows that $H_1(\Lambda_f \times_2 S^\infty, M \times P^\infty; Z_{\Psi(f)})$ has a subgroup which is isomorphic to $H_2^{(+)}(V, M; Z_2)$. On the other hand, since $(\overline{x}, e, m)^2 = (e, \partial\overline{x}, 1)$ the image of $(\overline{x}, e, m)$ in $H_1(\Lambda_f \times_2 S^\infty, M \times P^\infty; Z_{\Psi(f)})$ is of order 2. Hence

$$H_1(\Lambda_f \times_2 S^\infty, M \times P^\infty; Z_{\Psi(f)}) \approx H_2^{(+)}(V, M; Z_2) + Z_2 \approx H_2(V, M; Z_2).$$

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BEIJING (21) CENTURY MARKET RESEARCH CO., EVERYGREEN GARDEN, ZHONGSHAN PARK, BEIJING P.R. CHINA, P.C. 100031